Sensitivity of Lagrangian coherent structure identification to flow field resolution and random errors

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The effect of spatial and temporal resolutions and random errors on identification of Lagrangian coherent structures (LCSs) from Eulerian velocity fields is evaluated using two canonical flows: a two-dimensional vortex pair and a vortex ring formed by transient ejection of a jet from a tube. The flow field for the vortex pair case was steady and obtained analytically while the transient vortex ring flow was simulated using computational fluid dynamics. To evaluate resolution and random error effects, the flow fields were degraded by locally smoothing the flow and sampling it on a sparser grid to reduce spatial resolution, adding Gaussian distributed random noise to provide random errors, and/or subsampling the time series of vector fields to reduce the temporal resolution (the latter applying only for the vortex ring case). The degradation methods were meant to emulate distortions and errors introduced in common flow measurement methods such as digital particle image velocimetry. Comparing the LCS corresponding to the vortex boundary (separatrix) obtained from the degraded velocity fields with the true separatrix (obtained analytically for the vortex pair case or from high resolution, noise-free velocity fields for the vortex ring case) showed that noise levels as low as 5–10% of the vortex velocity can cause the separatrix to significantly deviate from its true location in a random fashion, but the “mean” location still remained close to the true location. Temporal and spatial resolution degradations were found to primarily affect transient portions of the flow with strong spatial gradients. Significant deviations in the location of the separatrix were observed even for spatial resolutions as high as 2% of the jet diameter for the vortex ring case. © 2010 American Institute of Physics. [doi:10.1063/1.3276062]

Lagrangian coherent structures (LCSs) represent generalized separatrices for unsteady fluid flows and as such are enormously useful for elucidating and quantifying complex transport processes in a wide variety of flows ranging from mixing in ocean currents to feeding of jellyfish.1–8 Moreover, identifying LCS as ridges in the finite time Lyapunov exponent (FTLE) field9 allows them to be determined from Eulerian data readily available from experiments [e.g., digital particle image velocimetry (DPIV) data] or numerical computations. The velocity information obtained from many experimental methods (DPIV included) and computational schemes, however, may have limited spatial or temporal resolution and/or may include random errors that are small but significant. Unfortunately, the influence of these effects on LCS is not readily apparent from the LCSs themselves as the computation of the FTLE field can produce sharp, well-defined ridges even for low quality data. Hence, the present investigation seeks to directly assess the influence of spatial and temporal resolutions and random errors on LCS identification by systematically degrading the velocity fields for two canonical flows and quantifying the effect on the LCS location as compared with the true location.

I. INTRODUCTION

LCSs can be identified as ridges in the (maximum) FTLE field for a fluid flow.10,9 For sharp ridges in a FTLE field obtained by long integration time T, the LCSs correspond to material surfaces9 that either attract (for backward time integration) or repel (for forward time integration) particle paths. Hence, LCSs represent generalized unstable and stable manifolds for unsteady flows. Due to their unique status as separatrices that attract or repel particle trajectories, LCSs have been employed to analyze a variety of fluid flow phenomena including understanding the mixing processes in Monterey Bay,1,2 investigating mixing and feeding induced by jellyfish swimming,7,3 identification of vortex boundaries required for entrainment calculations in unsteady flows,8,4 and for computing forces in aquatic locomotion.5,6

In studies employing LCS, the underlying flow field from which LCSs are extracted is usually taken as a given, obtained either from experimental measurements, computational fluid dynamics (CFD) simulations, or analytical equations. With the underlying velocity field \( \mathbf{V} \) known, the trajectory of a fluid particle at position \( \mathbf{x}_0 \) at time \( t_0 \) may be determined from

\[
\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{V}(\mathbf{x}(\tau)) \, d\tau
\]
\[ \dot{x}(t; t_0, x_0) = V(x(t; t_0, x_0), t), \]  
(1)

where \( x(t; t_0, x_0) = x_0 \). Integrating Eq. (1) provides a flow map \( d^{\text{T}}_{x_0}(x_0) \) for the position at time \( t = t_0 + T \) of the fluid particle initially at \( x_0 \) at time \( t_0 \) namely,

\[ d^{\text{T}}_{x_0}(x_0) = x(t_0 + T; t_0, x_0). \]

Then the (maximum) FTLE is defined as

\[ \sigma^T_{x_0}(x) = \frac{1}{|T|} \ln \lambda_{\text{max}}, \]

where \( \lambda_{\text{max}} \) is the maximum eigenvalue of

\[ (V d^{\text{T}}_{x_0}(x))^* (V d^{\text{T}}_{x_0}(x)), \]

and \( (\cdot)^* \) denotes the adjoint operation. In computing \( \sigma^T_{x_0}(x) \), a grid of tracer particles is advected according to Eq. (1) over a time \( T \). Owing to the exponential separation of tracer particles near separatrices, a dense initial grid is required for accurate calculation of \( \sigma^T_{x_0}(x) \) (Ref. 10) and identification of a sharp ridge in the FTLE field.9 Additionally, as shown by Shadden et al.,9 fluid flux across a ridge is of order \( 1/|T| \), so a large integration time is required to ensure the identified LCSs are material lines.

While tracer particle grid density and integration time have been identified as key factors influencing the accuracy of LCS identification for a given flow field, the sensitivity of LCS identification to errors in the underlying flow field is less well understood. Haller11 showed that as long as velocity field errors remain small in a time-weighted sense (i.e., they either have small magnitude or short duration), LCS identification is robust (i.e., relatively insensitive) to velocity field errors. Lermusiaux et al.2 and Lermusiaux1 present data apparently confirming Haller’s analysis, showing uncertainty (random error) in the velocity field translates to uncertainty in the FTLE field primarily in regions away from the LCS, although the authors do not describe how one might apply this result to predict the effect of errors in experimental data on LCS identification. Moreover, neither Haller11 nor the results of Lermusiaux and co-workers12 provide an indication of how systematic errors in the velocity field affect LCS identification. Griffa et al.,12 on the other hand, showed that smoothing the velocity field with a Gaussian weighted filter produced progressively larger errors in Lagrangian particle trajectories as the filter width became larger. While Griffa et al.12 were not interested in LCS, their results hint that LCS identification may be affected by velocity field smoothing (systematic error) since particle trajectory calculation is fundamental to determining the FTLE field.

The present study aims to clarify and quantify how errors in the underlying velocity field influence LCS identification from the FTLE field. The primary velocity field errors of interest are those associated with experimental velocity field measurement such as DPIV, namely, random errors and velocity field spatial and temporal resolutions. Here random errors are attributable to background noise in the measurement technique and the magnitude is typically no more than a few percent of the maximum measured flow speed. In the context of DPIV, the flow field spatial resolution is determined by the interrogation window size used in the cross-correlation algorithm and the magnification of the imaging optics. One velocity vector is obtained for each interrogation window, tiled over the flow image, and each vector represents the average velocity for the fluid within the interrogation window. To obtain only independent measurements, interrogation windows are typically allowed to overlap no more than 50% when processing the images. Hence, if the optics provides close-up (high magnification) images of the flow field, the vector spacing for a given interrogation window size is small compared with the flow features and there is little flow velocity variation over the extent of the interrogation window so the vectors accurately represent local flow measurements. If, on the other hand, the optics provides a zoomed-out view (low magnification) in order to capture a more global view of the flow, then the resulting vector field resolution may be sparse compared with the scale of flow features and the local averaging over the extent of the interrogation window will produce an artificially smoothed vector field. In the temporal domain, resolution is determined by the frame rate of the imaging camera relative to the relevant time scale of the flow evolution. In the following we consider the effects of random noise, flow field spatial resolution/smoothing, and temporal resolution on LCS identification by systematically degrading the velocity fields for two canonical flow fields (one analytical and one obtained from CFD) where the true LCSs are known.

II. MODEL FLOW FIELDS

The canonical flow fields considered in this study were a two-dimensional (2D) point vortex pair embedded in a uniform flow field and an axisymmetric vortex ring generated by transient ejection of a jet from a tube. Schematics of the flow field topology for the two cases are shown in Fig. 1. The flow field for the vortex pair was constructed analytically while the vortex ring flow was computed using CFD. Figure 1(b) includes a schematic of the computational domain and boundary conditions used in the CFD simulation. These flows were selected because they have simple and similar topologies, yet constitute the core feature of a wide variety of practical flows. The vortex pair problem represents a steady flow and has the advantage that the LCS (separatrix) is known analytically, while the vortex ring is an unsteady flow and allows the evaluation of transient effects on LCS identification.

The stream function for the vortex pair problem is obtained from potential flow theory as

\[ \psi = \frac{\Gamma}{2\pi} \ln \sqrt{\left(\frac{x}{D_p}\right)^2 + \left(\frac{y}{D_p} - 1/2\right)^2} - \frac{\Gamma}{2\pi D_p}, \]

(5)

where \( D_p \) is the separation between the vortices and \( \Gamma \) is the circulation magnitude (strength) of the vortices. The associated velocity field follows directly using the definition of \( \psi \). Likewise, the separatrix corresponding to the LCS for this flow is the stagnation streamline (\( \psi = 0 \)) and is described implicitly by
\[
\frac{1}{D_p} \left( \frac{x}{D_p} \right)^2 = \frac{e^{2(y/D_p)}(y/D_p - 1/2)^2 - (y/D_p + 1/2)^2}{1 - e^{2(y/D_p)}}.
\]

For computation of the FTLE field of the ideal (i.e., non-degraded) flow, the velocity field for this case was evaluated on a square grid of resolution 0.01/D_p.

The flow evolution for the vortex ring problem was simulated using the axisymmetric, unsteady, incompressible Navier–Stokes equations with zero swirl, namely,

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{\partial u_r}{\partial r} - \frac{1}{\rho} \frac{\partial \rho}{\partial r} = \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_r}{\partial r} \right) - \frac{\partial^2 u_r}{\partial x^2} \right],
\]

\[
\frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + u_x \frac{\partial u_x}{\partial x} = -\frac{1}{\rho} \frac{\partial \rho}{\partial x} + \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_x}{\partial r} \right) - \frac{\partial^2 u_x}{\partial x^2} \right],
\]

where \( u_r \) is the radial velocity component, \( u_x \) is the axial velocity component, \( \rho \) is the fluid pressure, \( \rho \) is the fluid density, and \( \nu \) is the kinematic viscosity of the fluid. The governing equations were solved using a finite volume method with a fixed time step. The solver scheme was second order accurate for both time and space. The SIMPLE algorithm was used for pressure-velocity coupling and the QUICK scheme was used for discretization of the convective terms.

The boundary conditions implemented in the CFD simulation are shown in Fig. 1(b). The “outflow” boundary condition corresponds to zero velocity gradient normal to the boundary and fixed pressure. The boundaries were selected far from the nozzle exit plane in order to minimize the effect of the outflow boundary condition on the flow evolution. The flow was driven by the inlet velocity \( U_p(t) \) for the tube. The inlet velocity was uniform across the tube inlet and varied in time according to a trapezoidal velocity program. Specifically, \( U_p(t) \) ramped up linearly to a specified velocity \( U_0 \) during a period \( 0.1t_p \), remained constant at \( U_0 \) for \( 0.8t_p \), and then returned to zero velocity linearly during the final \( 0.1t_p \) of the pulse, where \( t_p \) is the pulse duration. The pulse duration and \( U_0 \) were selected to give a stroke-to-diameter ratio, \( L/D \), of 2.0 and jet Reynolds number, Re, of 1000 where \( L/D \) and Re are defined according to

\[
\frac{L}{D} = \frac{1}{D} \int_0^t U_p(t) dt
\]

and

\[
Re = \frac{U_p D}{\nu}.
\]

The flow was simulated on a uniform grid of quadrilateral cells with side length 0.01D for a total mesh size of 1500×350 \( (N_r \times N_x) \). The uniform time step used was \( \Delta \tau = \Delta t U_p/D = 0.02 \). From previous work using the same solution method, it is known that the specified mesh and time resolution reproduce the vortex ring bulk properties such as hydrodynamic impulse and advected fluid volume (in the vortex ring boundary) to within 1%. The flow was simulated for a total duration of \( t^* = 6 \) where \( t^* = t/t_p \).

To illustrate the flow evolution, the velocity and vorticity fields for \( t^* = 0.5, 1.5, \) and 3.0 are shown in Fig. 2. Only every tenth vector is shown in the velocity fields for clarity and a vector of scale \( U_0 \) is shown in the upper right-hand corner of the velocity field frames. The vorticity contours
were set according to the peak vorticity level for each frame, as indicated in the figure caption, in order to facilitate display of the vorticity evolution across most of the flow evolution. As illustrated by the flow fields in Figs. 2(a) and 2(d), during jet ejection the boundary layer on the tube wall separates at the tube exit and begins rolling up into a vortex. After the jet terminates [\( \tau^t > 1 \), Figs. 2(b) and 2(e)], the vorticity flux from the tube ceases and the leading vortex ring entrains the remainder of the shear layer. Finally, the flow relaxes to a nearly steadily translating vortex ring by \( \tau^t \approx 3 \) [Figs. 2(c) and 2(f)]. The vortex ring speed and diameter at this point are given by \( W_r \) and \( D_r \), respectively (as illustrated in Fig. 1). For the case simulated, \( W_r = 0.47U_0 \) and \( D_r = 1.35D \) at \( \tau^t = 3 \) (based on the peak vorticity location).

For comparison purposes, it is helpful to note that \( W \) and \( U_0 \) are of the same order. Using the slug model to estimate the vortex ring velocity and circulation yields

\[
W_r = \frac{\Gamma_r}{L/D},
\]

where \( \Gamma_r \) is the vortex ring circulation. For \( L/D = 2 \), \( W_r \approx \Gamma_r/(2D) \), or \( W_r \approx \pi W \) for matched circulation and ignoring the difference between \( D_r \) and \( D \).

### III. LCS IDENTIFICATION AND SENSITIVITY ANALYSIS

The FTLE fields in this study were calculated by using a software package called MANGEN developed by Francois Lekien and Chad Coulliette in 2001 (http://www.lekien.com/~francois/software/mangen/). MANGEN determines the FTLE field by computing \( \sigma^T_{0}(x) \) from Eq. (3) for a grid of massless particles placed in the domain and advected using the supplied velocity field(s). For the problems considered in this investigation, computation of \( \sigma^T_{0}(x) \) was performed with a uniform grid of massless particles with resolution of 0.005\( D \) for the vortex ring problem and 0.01\( D_p \) for the vortex pair problem. The integration for the vortex pair problem used a time step of 0.05\( D_p / W \) for 50 frames, corresponding to a total integration time of \( T = 2.5D_p / W \). Integration for the vortex ring problem was computed for a total duration of \( |T'| = |T/t_p| = 3 \). The grid density and integration times chosen produced sharp ridges for both the attracting and repelling LCSs and ensured they were very nearly material surfaces (see Sec. I). The domain size used in the FTLE computation was \( 6D \times 1.5D \) in the \( x-r \) plane for the vortex ring problem and \( 2.5D_p \times 3.5D_p \) in the \( x-y \) plane for the vortex pair problem.

The vortex boundary (separatrix) for the vortex pair problem is given by Eq. (6), but it may also be identified by combining the outermost ridges of the attracting and repelling LCSs as in Shadden et al.\(^5\) This is illustrated in Fig. 3(a) where the vortex boundaries from both the LCS and Eq. (6) are compared. The agreement is very good and motivates using LCS for identifying the “true” separatrix for the vortex ring problem as well. The vortex ring separatrix determined in this way is illustrated in Fig. 3(b) where only the outermost LCS ridges are plotted for \( \tau^t = 3 \). The axial coordinates in Fig. 3(b) are with respect to \( x_r \) which is the \( x \) location of the ring peak vorticity at \( \tau^t = 3 \). In the following, the term “true separatrix” refers to the analytical separatrix given by Eq. (6) for the vortex pair case and the boundary determined from the outermost LCS ridges computed from the non-degraded data [Fig. 3(b)] for the vortex ring case. LCS identification for degraded velocity fields will be evaluated by comparing the separatrices obtained from the outermost ridges of the attracting and repelling LCSs with the true separatrices. The separatrix (outermost LCS ridges) was chosen as the basis for evaluation of LCS identification because this is the dominant feature in the topology for these flows (indeed the only feature in the vortex pair case); it corresponds to the highest \( \sigma^T_{0}(x) \) values and as such it is less sensitive to changes in \( |T'| \), the density of the massless particle grid, or the order of the particular method used to compute \( \sigma^T_{0}(x) \); and the amount of fluid contained in this boundary is relevant for studying entrainment and advection.

To determine the sensitivity of LCS identification to errors in the velocity field data, degraded velocity fields were generated, then \( \sigma^T_{0}(x) \) was recomputed for the degraded velocity fields, and finally the resulting LCSs for the vortex separatrices were compared with the true separatrices. The velocity fields were degraded with regard to noise and/or spatial resolution. For the vortex ring data, it was also degraded with regard to temporal resolution.

Degradation with regard to noise was accomplished by adding Gaussian distributed random noise to both velocity fields by computing LCSs and ensured they were very nearly material surfaces...
components for all velocity fields used in the integration for computing $\sigma_0^2(x)$. The magnitude of the noise (i.e., standard deviation) was specified at levels relevant for DPIV data, namely, 1%, 2%, and 5% of the velocity scale for the problem. The velocity scale used for setting the noise level was $U_0$ for the vortex ring case. For the vortex pair case, it was recognized that DPIV measurements in actual experiments would likely seek to resolve most of the flow features near the vortex cores (where the flow speed approaches infinity), so a higher velocity scale of $10W$ was selected to set the noise level for this problem. This is the speed of the flow approximately $0.1D_p$ away from each vortex center.

Spatial degradation of the velocity fields was performed in a way to mimic the effect of increased interrogation window size (relative to the scale of flow features) in DPIV data. For the vortex pair data, this was accomplished by analytically computing the local velocity average over a square area equal to twice the final (degraded) vector spacing (mimicking 50% interrogation window overlap) at locations corresponding to the desired vector spacing (resolution). For the vortex ring case, the original high resolution CFD data were used to produce the degraded velocity fields by numerically averaging over a square area equal to twice the desired degraded vector spacing (resolution) and then subsampling the data to remove the intervening vectors. Notice that in neither case does this operation account for the bias toward lower velocities in DPIV data processing when there is significant shear across the interrogation window. For cases where both noise and spatial resolution degradation were employed, the noise was added after the spatial degradation was performed. Resolutions used for the degraded velocity fields in the vortex ring case were $0.02D_p$, $0.07D_p$, and $0.12D_p$. For the vortex pair case, the resolutions analyzed were $0.025D_p$, $0.0625D_p$, and $0.125D_p$. Finally, temporal resolution degradation for the vortex ring case was accomplished by subsampling the set of vector fields used in the integration to obtain $\sigma_0^2(x)$. The resolutions considered were $\Delta t = 0.04$, 0.08, and 0.16. Reduced temporal resolution was not combined with noise in any of the data sets because, as will be shown, the addition of noise had minimal effect on the mean LCS location.

Once the LCS for the separatrices were obtained from the degraded velocity fields, the effect of the degradation on LCS identification was assessed by finding the difference in the location of the forward and rear saddle points ($s_F$ and $s_R$, respectively, as shown in Fig. 1) and the difference in area within the identified vortex ring boundary as compared with the true separatrix. For the vortex ring case, all comparisons were done at $r^* = 3$. Also, the location of the saddle points could not be determined directly in the vortex ring case because the computed LCS never intersected the axis for this axisymmetric problem. Instead, the location of the saddle points was estimated by fitting a cubic polynomial (constrained to have zero slope and curvature with the appropriate sign at the axis) to the attracting and repelling LCSs and finding the intersection of this curve with the axis. Based on the true separatrix computed from the high resolution data, it is estimated that this method for determining the locations of $s_F$ and $s_R$ is accurate to within $\pm 0.005D$.

IV. RESULTS AND DISCUSSION

To illustrate the effects of noise and spatial degradation on the computed LCS, the separatrices determined from the outermost ridge of the attracting and repelling LCSs are shown in Figs. 4 and 5 for the vortex pair and vortex ring cases, respectively. Each figure shows four cases corresponding to high resolution with low noise, high resolution with high noise, low resolution with low noise, and low resolution with high noise, where “high” and “low” correspond to the highest and lowest values used for degraded velocity fields (e.g., $0.02D$ and $0.12D$ for the high and low resolutions of the vortex ring case). In these figures, the dashed line indicates the true location of the separatrix (determined analytically for the vortex pair case and from the original high resolution CFD simulation results for the vortex ring case), and the solid line indicates the separatrix determined from the outer ridges of the LCS computed from the degraded velocity fields. In Fig. 5, the separatrices do not extend all the way to the axis because near the axis the FTLE field had a low value and fell below the threshold value used for finding the ridge in the FTLE field.

Several qualitative observations immediately follow from Figs. 4 and 5. For the vortex pair case (Fig. 4) it is clear that noise has the largest effect, in part because of the relatively large noise values used in this case compared to those used in the vortex ring case. (The velocity scale for noise in the vortex pair case $10W$ is $\sim 3$ times larger than the velocity scale used in the vortex ring case $U_0$, as discussed in Sec. II.) The noise introduces deviations in the location of the sepa-
ratrix, but these deviations are more or less evenly distributed on either side of the true separatrix so that the “mean” separatrix position falls near the true location. Changes in resolution/smoothing appear to have a very little effect for the vortex pair case except to change the scale of the deviations introduced by the noise. The vortex ring case (Fig. 5) shows a very little effect from noise, but reducing the resolution which simultaneously increases smoothing has a marked effect, especially on the front edge of the computed LCS. As the resolution is decreased, even just a small amount (see the 0.02D resolution case), the separatrix appears noticeably smaller and somewhat distorted compared with the true separatrix.

To quantify the above observations, the locations of the forward and rear saddle points on the separatrices and the area bounded by the separatrices were determined as described previously. Figure 6 shows the mean location of the forward and rear saddle points relative to the true value.

FIG. 5. LCS results for degraded velocity fields from the vortex ring case (all at \( t = 3 \)): (a) 0.02D resolution with 1% noise (0.01\( U_0 \)), (b) 0.12D resolution with 1% noise (0.01\( U_0 \)), (c) 0.02D resolution with 5% noise (0.05\( U_0 \)), and (d) 0.12D resolution with 5% noise (0.05\( U_0 \)). The dashed lines indicate the true separatrix and the solid lines indicate the separatrix determined from the LCS. Representative cases chosen from multiple realizations are shown for each set of parameters.

FIG. 6. Mean saddle point Locations \( \bar{s}/D \) relative to their true locations \( s_0 \): (a) vortex pair case and (b) vortex ring case. The legend indicates the noise level for the lines and dashed lines correspond to front saddle points.
(\bar{s} - s_0) for the vortex pair and vortex ring cases. The means for the vortex pair case were taken over ten realizations of the noise contribution to the velocity field, while the vortex ring means were computed from six realizations. [It should be noted that the 0.01D resolution in Fig. 5(b) corresponds to LCS computed from the full-resolution vector field degraded only with noise.] The mean saddle point locations for the vortex pair case are not significantly affected by either noise or resolution effects [Fig. 6(a)]. Except for two cases, all of the errors are approximately 0.005D_p, which is half the resolution of the grid used to compute the FTLE field, and one would not expect results significantly better than this, even under ideal circumstances. The two cases with errors greater than 0.01D_p are for high noise, low resolution cases and may be the result of an insufficiently large sample to remove all random variations. The standard deviation (\sigma) of the saddle point locations, however, does depend significantly on the noise level, as shown in Fig. 7. There is also a weak dependence of the standard deviation on resolution, with lower resolution corresponding to higher \sigma. So, even though the location of the saddle points for the vortex pair case may not depend significantly on resolution or noise on average, significant variations do exist for individual realizations and a larger number of realizations will be required for larger noise and/or lower resolution levels in order to obtain reliable results for a mean. The saddle point locations for the vortex ring case [Fig. 6(b)] also show almost no significant dependence on noise. This is due in part to the lower noise levels used for this case, the small effect of noise for this problem [see Fig. 9(b)], and the tendency for the least-squares polynomial fit used to locate the saddle points to average out the effects of noise on this metric. Because the least-squares fit distorts the statistics for the vortex ring saddle point locations, the standard deviation of the saddle point locations is not shown for this case. The vortex ring saddle point locations do depend significantly on resolution, however. The effect is mostly strongly observed for the front saddle point (s_F). Significantly, even a resolution of 0.02D (which is still a good resolution for DPIV results) shows a significant difference between s_F and the true location obtained from the 0.01D resolution results.

The results for the mean fluid area within the separatrix (A) relative to the true area (A_0) are shown in Figs. 8 and 9. [To handle the gap between the identified LCS and the axis in the vortex ring case for the low resolution results (see Fig. 5), the LCS was extended straight down to the axis for the area calculations.] Generally the results from Fig. 8 follow the observations made from the mean saddle point locations, namely, that the mean for the vortex pair case shows little dependence on either noise or resolution, while the vortex ring case has a very strong dependence on resolution, but not on noise. The exception is for 5% noise with the vortex pair case, which is significantly different from the rest. Apparently, once the noise becomes large enough, a significant increase in area is observed as a consequence of the larger area available outside the vortex boundary for the separatrix to expand. Of course, the dependence of the area on resolution in the vortex ring case is due primarily to the dependence of the attracting (i.e., front) LCS on the resolution of the underlying velocity field. As with Fig. 8, Fig. 9 shows trends similar to those expected from the saddle point location results. Specifically, the standard deviation of results
tends to increase with noise level and with reduced resolution, where the dependence on the former is stronger. It should be noted that the vortex pair results in Fig. 9(a) are more or less the direct extension of the vortex ring results to higher noise levels owing to the scaling of noise by $10W$ for the vortex pair case and by $U_0$ for the vortex ring case. Consequently, the vortex pair results have $2–5$ times higher standard deviations at corresponding noise percentages.

The observation that noise has a minimal effect on the mean location of the saddle points and on the mean value of the area within the separatrix is qualitatively consistent with Haller’s conclusion that LCS identification is robust as long as velocity field errors remain small in a time-weighted sense. That is, even though random variations in the velocity field may significantly affect the trajectories of individual Lagrangian particles, the LCS still ends up in more or less the correct location (on average). This is not to say that the effects of random variations on LCS identification are insignificant. The area results for the vortex pair case with $5\%$ noise (noise standard deviation of $0.5W$) show that significant deviations can occur if the noise level is large enough. Moreover, individual realizations of LCSs may be significantly distorted by noise, as indicated by the standard deviation of the saddle point and area results (see Figs. 7 and 9). This is probably best illustrated by computing the area between the true separatrix and the separatrix determined from the degraded velocity fields ($\delta A$). The mean of this quantity as a fraction of the true area ($A_0$) is shown in Fig. 10 for the vortex pair case. Clearly there are significant gaps between the true and computed separatrices as the noise level increases, and the magnitude of the gaps is amplified by reducing the resolution. Nevertheless, the relative independence of the vortex pair results for $A$ and $\bar{\sigma}$ on resolution suggests that smoothing the velocity field may be an effective way of increasing the reliability of LCS identification for especially noisy data that do not involve strong velocity gradients [the exception being the attracting (front) LCS for the vortex ring case, which is discussed further below]. Considering spatial resolution effects, on the other hand, the strong dependence of the vortex ring results (for both area and saddle point location) on spatial resolution is consistent with the results of Griffa et al., since reducing spatial resolution includes increased smoothing of the velocity field in the present analysis (although Griffa et al. studied particle paths and not LCS directly).

Despite the qualitative agreement of some of the present results with prior studies, the distinctly different dependence of LCS identification on reduced resolution for the vortex pair and vortex ring cases warrants further discussion. To address this issue, consider the saddle point locations. For the vortex pair case, reducing the resolution by smoothing and subsampling the velocity field should not significantly affect the flow field near the saddle points because they reside where the velocity gradient (in the $x$ direction) is nearly linear. Hence, smoothing will have a little effect on the location of the saddle points unless the interrogation window is very large (order $D_p$ or bigger). Similarly, the repelling (i.e., rear) LCS for the vortex ring case is obtained by integrating from $t^*=3$ to $t^*=6$ (for a total integration time of $T^*=3$).

FIG. 9. Standard deviation ($\sigma$) of the separatrix area: (a) vortex pair case and (b) vortex ring case. The noise levels are as a fraction of $10W$ for the vortex pair case and $U_0$ for the vortex ring case.

FIG. 10. Mean of the area gap between the true and computed separatrices ($\bar{\delta}A$) relative to the total area within the true separatrix ($A_0$) for the vortex pair case.
During this period the vortex ring is steady in a frame of reference moving with the vortex ring, so the arguments applied to the vortex pair case are applicable here also, which is why the location of the rear saddle point is not strongly affected by resolution for the vortex ring case. The attracting LCS, on the other hand, is obtained by integrating backward in time from \( t^* = 3 \) to \( t^* = 0 \), which includes the unsteady formation of the vortex ring. It is also during this period that much stronger velocity gradients occur, especially near the nozzle exit plane during jet initiation, which are much more strongly affected by the smoothing effect employed when reducing the resolution in the present investigation, and thereby explaining the strong dependence of the attracting LCS location on resolution. Apparently, systematic errors in the velocity field can have a strong influence on LCS identification if they significantly alter velocity gradients used in the FTLE computation.

The results for degrading the time resolution of the vortex ring case are shown in Figs. 11 and 12. Figure 11 shows the comparison (at \( t^* = 3 \)) between the true separatrix and the separatrices determined from lower time resolution with full spatial resolution and no noise. As with spatial resolution, it appears that reducing temporal resolution primarily affects the front portion of the separatrix (attracting LCS). This is illustrated quantitatively in Fig. 12, which shows a small effect from time resolution on the location of the rear saddle point location, a strong effect on the front saddle point location, and an effect on the area within the separatrix, once the time resolution is low enough. The prominent effect of reduced temporal resolution on the attracting LCS portion of the separatrix is analogous to the effect of reduced spatial resolution, emphasizing the need to properly resolve the spatial and temporal aspects of the unsteady, high-gradient portions of the flow for adequate LCS identification.

V. CONCLUSIONS

The effect of noise and spatial resolution on LCS identification for the separatrices of a 2D vortex pair and an axisymmetric vortex ring has been considered. The unsteady nature of the vortex ring formation also allowed the effect of temporal resolution to be evaluated. Very little effect from noise on LCS identification was observed for the low noise levels considered in the vortex ring case. The higher noise levels evaluated in the vortex pair case (~3 times higher)
showed a significant effect on individual realizations of the LCSs defining the separatrix (in terms of saddle point locations, total area within the separatrix, and area between the degraded and true separatrices), but the mean location of the LCS was near the true location when several realizations were considered. This result indicates that a significant statistical sample is required for reliable LCS identification when large random disturbances are present in the data. Reduced spatial resolution had a weaker effect on the vortex pair case, tending to increase the scale of the variations introduced by noise in individual realizations of the separatrix. For the vortex ring case, on the other hand, reduced spatial and temporal resolutions had a strong effect on the location of the attracting LCS, but only a weak effect on the repelling LCS. It was argued that the large spatial gradients and unsteadiness in the flow initiation accounted for the sensitivity of the attracting LCS to temporal and spatial resolutions since the methods for degrading these in the present study amounted to spatial and temporal smoothing of the flow. Clearly, systematic measurement errors can have a significant effect on LCS identification as well, and care must be exercised when strong spatial or temporal gradients exist in the flow.